SOLVED EXAMPLES

Ex.1 $f(x) = 2x^3 - 21x^2 + 36x + 7$ has a maxima at -

(A) x = 2 (B) x = 1 (C) x = 6 (D) x = 3

Sol. $f'(x) = 6x^2 - 42x + 36$

$f''(x) = 12x - 42$

Now $f'(x) = 0 \Rightarrow 6(x^2 - 7x + 6) = 0$

$\Rightarrow x = 1, 6$

Also $f''(1) = 12 - 42 = -30 < 0$

$\therefore$ f(x) has a maxima at x = 1 **Ans.[B]**

Ex.2 The minimum value of the function x^x (x > 0) is at -

(A) x = 1 (B) x = e

(C) $x = e^{-1}$ (D) None of these

Sol. Let $y = x^x \Rightarrow \log y = x \log x$

$\Rightarrow \frac{d}{dx}(\log y) = 1 + \log x$

and $\frac{d^2}{dx^2}(\log y) = \frac{1}{x} = x^{-1}$

Now for minimum value of y or log y

$\frac{d}{dx}(\log y) = 0 \Rightarrow 1 + \log x = 0$

$\Rightarrow x = e^{-1}$ Again for $x = e^{-1}$

$\frac{d^2}{dx^2}(\log y) = e > 0$

$\Rightarrow$ y is minimum at $x = e^{-1}$ **Ans.[C]**

Ex.3 If x = p and x = q are respectively the maximum and minimum points of the function $x^5 - 5x^4 + 5x^3 - 10$, then -

(A) p = 0, q = 1 (B) p = 1, q = 0

(C) p = 1, q = 3 (D) p = 3, q = 1

Sol. Let $f(x) = x^5 - 5x^4 + 5x^3 - 10$, then

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$= 5x^2(x-1)(x-3)$

and $f''(x) = 20x^3 - 60x^2 + 30x$

For maxima and minima

$f'(x) = 0 \Rightarrow 5x^2(x-1)(x-3) = 0$

$\Rightarrow x = 0, 1, 3$ Also $f''(1) = -10 < 0$

$\Rightarrow$ x = 1 is a point of maxima $\Rightarrow$ p = 1

and $f''(3) = 90 > 0$

$\Rightarrow$ x = 3 is a point of minima $\Rightarrow$ q = 3.**Ans.[C]**

Ex.4 Let x, y be two variables and x > 0, xy = 1. Then minimum value of x + y is -

(A) 1 (B) 2 (C) 3 (D) 4

Sol. Let A = x + y = x + 1/x ($\because$ xy = 1)

$\Rightarrow \frac{dA}{dx} = 1 - \frac{1}{x^2}, \frac{d^2A}{dx^2} = \frac{2}{x^3}$

Now $\frac{dA}{dx} = 0 \Rightarrow x = 1, -1$

Also at x = 1, $\frac{d^2A}{dx^2} = 2 > 0$

x = 1 is a minimum point of A. So minimum value of A = 1 + 1/1 = 2. **Ans.[B]**

Ex.5 The maximum value of function sin x (1+ cos x) occurs at -

(A) x = π/4 (B) x = π/2

(C) x = π/3 (D) x = π/6

Sol. Let $f(x) = \sin x(1 + \cos x) = \sin x + \frac{1}{2}\sin 2x$,

then $f'(x) = \cos x + \cos 2x$

and $f''(x) = -\sin x - 2\sin 2x$

For maximum value $f'(x) = 0$

$\Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow \cos x = -\cos 2x$

$\Rightarrow \cos x = \cos(\pi - 2x)$

$\Rightarrow x = \pi - 2x \Rightarrow x = \pi/3$

Again $f''(\pi/3) = -\sin(\pi/3) - 2\sin(2\pi/3)$

$= -\frac{3\sqrt{3}}{2} < 0$

$\Rightarrow$ Maximum value of function occurs at x = π/3 **Ans.[C]**

Ex.6 The maximum value of 3 sin x + 4 cos x is -

(A) 3 (B) 4 (D) 5 (D) 7

Sol. Let f (x) = 3 sin x + 4 cos x

$\Rightarrow f'(x) = 3\cos x - 4\sin x$

$f''(x) = -3\sin x - 4\cos x$

Now $f'(x) = 0 \Rightarrow 3\cos x - 4\sin x = 0$

$\Rightarrow \tan x = 3/4$

Also then sin x = 3/5, cos x = 4/5 and so at

$x = \tan^{-1}(3/4)$

$f''(x) = -3(3/5) - 4(4/5) < 0$

$\Rightarrow$ f(x) has a maxima at tan x = 3/4. Also its maximum value

$= 3(3/5) + 4(4/5) = 5$ **Ans.[C]**

Ex.7 If x = –1 and x = 2 are extreme points of the function $y = a \log x + bx^2 + x$, then-

(A) a = 2, b = 1/2 (B) a = 2, b = – 1/2

(C) a = – 2, b = 1/2 (D) a = – 2, b = – 1/2

Sol. $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

Since x = –1 and x = 2 are extreme points so dy/dx at these points must be zero. So

$-a - 2b + 1 = 0$ and $a/2 + 4b + 1 = 0$

$\Rightarrow a + 2b - 1 = 0$ and $a + 8b + 2 = 0$

$\Rightarrow a = 2, b = -1/2$ **Ans.[B]**

Ex.8 In [0, 2π] one maximum value of x + sin 2x is -

(A) $\frac{2\pi}{3} + \frac{\sqrt{3}}{2}$ (B) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

(C) $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ (D) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$

Sol. Let f (x) = x + sin 2x

$\Rightarrow f'(x) = 1 + 2\cos 2x$

$f''(x) = -4 \sin 2x$

Now $f'(x) = 0 \Rightarrow \cos 2x = -1/2$

$\Rightarrow 2x = 2\pi/3, 4\pi/3, ...$

$\Rightarrow x = \pi/3, 2\pi/3$

But $f''(\pi/3) = -4(\sqrt{3}/2) < 0$

$\therefore$ f(x) is maximum at x = π/3 and its one maximum value

$= \pi/3 + \sin(2\pi/3)$

$= \pi/3 + \sqrt{3}/2$ **Ans.[C]**

Ex.9 The maximum and minimum values of sin 2x – x are-

(A) 1, –1 (B) $\frac{3\sqrt{3}-\pi}{6}, \frac{\pi-3\sqrt{3}}{6}$

(C) $\frac{\pi-3\sqrt{3}}{6}, \frac{3\sqrt{3}-\pi}{6}$ (D) Do not exist

Sol. f(x) = sin 2x – x

$f'(x) = 2\cos 2x - 1$

$f''(x) = -4\sin 2x$

Now $f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0$

$\Rightarrow x = n\pi \pm \pi/6$ n = 0, 1,2, ...

$\Rightarrow x = \pi/6, 5\pi/6, 7\pi/6, -\pi/6, ...$

But $f''(\pi/6) = -2\sqrt{3} < 0$

$\Rightarrow x = \pi/6$ is a max. point

Also $f''(5\pi/6) = 2\sqrt{3} > 0$

$\Rightarrow x = 5\pi/6$ is a min. point

Hence one max. value = $f(\pi/6) = \frac{3\sqrt{3}-\pi}{6}$

one min. value = $f(5\pi/6) = -\frac{3\sqrt{3}-5\pi}{6}$

But it is not there in given alternatives. Hence by alternate position another min. point is – π /6 so one min. value

$= f(-\pi/6) = \frac{\pi-3\sqrt{3}}{6}$ **Ans.[B]**

Ex.10 For what values of x, the function sinx + cos 2x (x>0) is minimum -

(A) $\frac{n\pi}{2}$ (B) $\frac{3(n+1)\pi}{2}$

(C) $\frac{(2n+1)\pi}{2}$ (D) None of these

Sol. Let f(x) = sin x + cos 2x, then

$f'(x) = \cos x - 2\sin 2x$

and $f''(x) = -\sin x - 4\cos 2x$

For minimum $f'(x) = 0 \Rightarrow \cos x - 4\sin x \cos x = 0$

$\Rightarrow \cos x(1 - 4\sin x) = 0$

$\Rightarrow \cos x = 0$ or $1 - 4\sin x = 0 \Rightarrow x = (2n+1)\pi/2$ or

$x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{4}\right), n \in Z$

Now $f''\left\{(2n+1)\frac{\pi}{2}\right\}$

$= -\sin\left\{(2n+1)\frac{\pi}{2}\right\} - 4\cos(2n+1)\pi$

$= -(-1)^n - 4(-1)^{2n+1} > 0$

The function is minimum at $x = \frac{(2n+1)\pi}{2}$

Ans.[C]

Ex.11 The minimum value of

64 sec x + 27 cosec x, 0 < x < π/2 is-

(A) 91 (B) 25

(C) 125 (D) None of these

Sol. Let y = 64 sec x + 27 cosec x

$\Rightarrow \frac{dy}{dx}$ = 64 sec x tan x– 27 cosec x cot x

$\frac{d^2y}{dx^2}$ = 64 $\sec^3$ x + 64 sec x $\tan^2$x + 27 cosec^3 x + 27 cosec x $\cot^2$ x

Now $\frac{dy}{dx} = 0 \Rightarrow$ 64sec x tan x =27 cosec x cot x

$\Rightarrow \tan^3 x = 27/64$

$\Rightarrow \tan x = 3/4$

Also then $\frac{d^2y}{dx^2} > 0$ $(\because 0 < x < /2)$

So y is minimum when

x = $\tan^{-1}$ (3/4) and its

min. value = 64 (5/4) + 27 (5/3) = 125 **Ans.[C]**

Ex.12 If $0 \le c \le 5$, then the minimum distance of the point (0, c) from parabola $y = x^2$ is-

(A) $\sqrt{c-4}$ (B) $\sqrt{c-1/4}$

(C) $\sqrt{c+1/4}$ (D) None of these

Sol. Let $(\sqrt{t}, t)$ be a point on the parabola whose distance from (0, c), be d. Then

$z = d^2 = t + (t-c)^2 = t^2 + t(1-2c) + c^2$

$\Rightarrow \frac{dz}{dt} = 2t + 1 - 2c, \frac{d^2z}{dt^2} = 2 > 0$

Now $\frac{dz}{dt} = 0 \Rightarrow t = c - 1/2$

which gives the minimum distance. So

min. distance = $\sqrt{(c-1/2)+(-1/2)^2}$

$= \sqrt{c-1/4}$ **Ans.[B]**

Ex.13 The minimum value of the function

$\frac{40}{3x^4+8x^3-18x^2+60}$ is -

(A) 2/3 (B) 3/2

(C) 40/53 (D) None of these

Sol. Let y = $\frac{1}{40}$ $(3x^4 + 8x^3 - 18x^2 + 60)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{40}(12x^3 + 24x^2 - 36x)$

and $\frac{d^2y}{dx^2} = \frac{1}{40}$ $(36x^2 + 48x - 36)$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 + 2x^2 - 3x = 0$

or x(x –1) (x + 3) = 0

or x = 0, 1, – 3

At x = 0, $\frac{d^2y}{dx^2} = -36 < 0$

$\therefore$ y is maximum at x = 0

$\Rightarrow$ the given function i.e. 1/y is minimum at x = 0

$\therefore$ minimum value of the function

$\frac{40}{60} = \frac{2}{3}$ **Ans.[A]**

Ex.14 If $\frac{dy}{dx} = (x-1)^3 (x-2)^4$, then y is -

(A) maximum at x = 1

(B) maximum at x = 2

(C) minimum at x = 1

(D) minimum at x = 2

Sol. $\frac{dy}{dx} = 0 \Rightarrow x = 1, 2$. If h > 0 is very small number, then

at x = 1–h , $\frac{dy}{dx}$ = (–) (+) = – ve

x = 1 + h, $\frac{dy}{dx}$ = (+) (+) = + ve

at x = 1, $\frac{dy}{dx}$ changes its sign from –ve to + ve which shows that x = 1 is a minimum. **Ans.[C]**

Ex.15 The maximum area of a rectangle of perimeter 176 cms. is -

(A) 1936 sq.cms. (B) 1854 sq.cms.

(C) 2110 sq.cms. (D) None of these

Sol. Let sides of the rectangle be x, y ; then

2x + 2y = 176 ...(1)

$\therefore$ Its area A = xy = x (88– x)

[form (1)] $= 88x - x^2$

$\Rightarrow \frac{dA}{dx} = 88 - 2x,\ \frac{d^2A}{dx^2} = -2 < 0$

Now $\frac{dA}{dx} = 0 \Rightarrow x = 44$;

Also then $\frac{d^2A}{dx^2} < 0$. So area will be maximum when x = 44 and maximum area

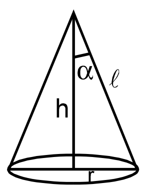
= 44 x 44 = 1936 sq. cms. **Ans.[A]**

Ex.16 The semivertical angle of a right circular cone of given slant height and maximum volume is-

(A) $\tan^{-1} 2$ (B) $\tan^{-1} (\sqrt{2})$

(C) $\tan^{-1}\left(\frac{1}{2}\right)$ (D) $\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Sol. Let ℓ be the slant height and α be the semivertical angle of the right circular cone. Also suppose that h and r are its height and radius of the base.



Then $h = \ell \cos\alpha$, $r = \ell \sin\alpha$

Now volume $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\ell^3 \sin^2\alpha \cos\alpha$

$\therefore \frac{dV}{d\alpha} = \frac{1}{3} = \pi\ell^3[-\sin^3\alpha + 2\sin\alpha\cos^2\alpha]$

$= \frac{1}{3}\pi\ell^3[-\sin^3\alpha + 2\sin\alpha(1 - \sin^2\alpha)]$

$= \frac{1}{3}\pi\ell^3[2\sin\alpha - 3\sin^3\alpha]$

$\therefore \frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi\ell^3[2\cos\alpha - 9\sin^2\alpha\cos\alpha]$

Now $\frac{dV}{d\alpha} = 0 \Rightarrow \sin\alpha = 0$ or $2 - 3\sin^2\alpha = 0$

Now $\alpha \neq 0$ $\therefore 2 = 3\sin^2\alpha$

or $2\sin^2\alpha + 2\cos^2\alpha = 3\sin^2\alpha$

or $\tan^2\alpha = 2 \Rightarrow \tan\alpha = \sqrt{2}$

When $\tan\alpha = \sqrt{2}$, $\frac{d^2V}{d\alpha^2} < 0$

Thus when $\alpha = \tan^{-1}\sqrt{2}$, volume will be maximum. **Ans. [B]**

Ex.17 Two parts of 10 such that the sum of the twice of first with the square of second is minimum, are-

(A) 9, 1 (B) 5, 5 (C) 4, 6 (D) 1, 9

Sol. Let two parts be x and (10–x). If

$y = 2x + (10 - x)^2$

Then $\frac{dy}{dx} = 2 - 2(10 - x) = 2x - 18$

Now $\frac{dy}{dx} = 0$ x = 9

Also then $\frac{d^2y}{dx^2} = 2 > 0$. Hence when x = 9, value of y is minimum. So required two parts of 10 are 9 and 1. **Ans.[A]**

Ex.18 For the curve $y = xe^x$ -

(A) x = 0 is a point of maxima

(B) x = 0 is a point of minima

(C) x = – 1 is a point of minima

(D) x = – 1 is a point of maxima

Sol. $y = xe^x \Rightarrow \frac{dy}{dx} = xe^x + e^x$

and $\frac{d^2y}{dx^2} = xe^x + 2e^x$

now $\frac{dy}{dx} = 0 \Rightarrow e^x(x + 1) = 0$

$\Rightarrow x = -1$ $[\because e^x > 0, \forall x]$

and at x = – 1, $\frac{d^2y}{dx^2} = e^{-x}(-1 + 2) > 0$

Therefore x = – 1 is a point of minima. **Ans.[C]**

Ex.19 If sin x – x cos x is maximum at x = nπ, then-

(A) n is an odd positive integer

(B) n is an even negative integer

(C) n is an even positive integer

(D) n is an odd positive or even negative integer

Sol. Let f(x) = sin x – x cos x, then

$\Rightarrow f'(x) = \cos x - \cos x + x\sin x = x\sin x$

$f''(x) = x\cos x - \sin x$

Now $f'(x) = 0 \Rightarrow x\sin x = 0$

$\Rightarrow x = 0, n\pi$ n = 0, 1,2, ...

Also $f''(n\pi) = n\pi\cos n\pi - \sin n\pi$

$= (-1)^n n\pi$

But f(x) is maximum at x = nπ when $f''(n\pi) < 0$

$\Rightarrow (-1)^n n\pi < 0 \quad \Rightarrow \quad (-1)^n n < 0$

⇒ either n is an odd positive or even negative integer. **Ans.[D]**

Ex.20 $x(1-x^2),\ 0 \le x \le 2$ is maximum at -

(A) x = 0 (B) x = 1

(C) $x = 1/\sqrt{3}$ (D) Nowhere

Sol. Let $y = x(1 - x^2)$

$\Rightarrow \frac{dy}{dx} = (1-x^2) - 2x^2 = 1 - 3x^2$

and $\frac{d^2y}{dx^2} = -6x$

Now $dy/dx = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

Now at $x = \frac{1}{\sqrt{3}}$, $\frac{d^2y}{dx^2} < 0$.

Therefore y is maximum at $x = \frac{1}{\sqrt{3}}$ **Ans.[C]**

Ex.21 A curve whose slope at (x,y) is $x^2 - 2x$, passes through the point (2,0). The point with greatest ordinate on the curve is-

(A) (0, 0) (B) (0, 4)

(C) (0, 4/3) (D) (0,3/4)

Sol. Here $\frac{dy}{dx} = x^2 - 2x$

$\Rightarrow y = \frac{1}{3}x^3 - x^2 + c$

Since the curve passes through the point (2,0), therefore

$0 = (8/3) - 4 + c \Rightarrow c = 4/3$

∴ equation of curve $y = \frac{1}{3}x^3 - x^2 + \frac{4}{3}$ and

$\frac{dy}{dx} = x^2 - 2x.\ \frac{d^2y}{dx^2} = 2x - 2$

Now $\frac{dy}{dx} = 0 \Rightarrow x = 0, 2$

But at x = 0, $\frac{d^2y}{dx^2} = -2 < 0$

Thus at x = 0, y = 4/3 is maximum. **Ans.[C]**

Ex.22 $f(x) = 1 + 2\sin x + 3\cos^2 x\ (0 \le x \le 2\pi/3)$ is-

(A) minimum at x = π/2

(B) maximum at $x = \sin^{-1}(1/\sqrt{3})$

(C) minimum at x = π/3

(D) minimum at $x = \sin^{-1}(1/3)$

Sol. $f'(x) = 2\cos x - 6\cos x \sin x$

$f''(x) = -2\sin x + 6\sin^2 x - 6\cos^2 x$

$= -2\sin x + 12\sin^2 x - 6$

Now $f'(x) = 0 \Rightarrow \cos x = 0$ and $\sin x = 1/3$

or $x = \pi/2$ & $x = \sin^{-1}(1/3)$

so $f''(\pi/2) = -2 + 12 - 6 > 0$

$f''\left(\sin^{-1}\frac{1}{3}\right) = \frac{-2}{3} + \frac{4}{3} - 6 < 0$

∴ f(x) is minimum at x = π/2. **Ans.[A]**

Ex.23 The minimum value of $e^{(2x^2-2x-1)\sin^2 x}$ is -

(A) e (B) 1/e (C) 1 (D) 0

Sol. Let $y = e^{(2x^2-2x-1)\sin^2 x}$

and $u = (2x^2 - 2x - 1)\sin^2 x$

Now $\frac{du}{dx}$

$= (2x^2 - 2x - 1)\, 2\sin x \cos x + (4x - 2)\sin^2 x$

$= \sin x\,[2(2x^2 - 2x)\cos x + (4x - 2)\sin x]$

$\frac{du}{dx} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$

$\frac{d^2u}{dx^2} = \sin x \frac{d}{dx}\,[2(2x^2-2x-1)\cos x$

$+ (4x-2)\sin x] + \cos x\,[2\cos x(2x^2-2x-1)$

$+ (4x-2)\sin x]$

At x = nπ,

$\frac{d^2u}{dx^2} = 0 + 2\cos^2 n\pi(2n^2\pi^2 - 2n\pi - 1) > 0$

Hence at x = nπ, the value of u and so its corresponding the value of y is minimum and minimum value = $e^0 = 1$. **Ans.[C]**