MATRICES

(KEY CONCEPTS & SOLVED EXAMPELS)

══MATRICES══

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KEY CONCEPTS

1. Definition

A rectangular arrangement of numbers in rows and columns, is called a Matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. A matrix is represented by capital letters A, B, C etc. and its element are by small letters a, b, c, x, y etc.

2. Order of a Matrix

A matrix which has m rows and n columns is called a matrix of order m × n.

A matrix A of order m × n is usually written in the following manner-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ..a_{1j} & ..a_{1n} \\ a_{21} & a_{23} & a_{23} & ..a_{2j} & ..a_{2n} \\ ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ..a_{ij} & ..a_{in} \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ..a_{mj} & ..a_{mn} \end{bmatrix} \text{ or}$$

$A = [a_{ij}]_{m \times n}$ where $\begin{matrix} i = 1, & 2, ...m \\ i = 1, & 2, ...n \end{matrix}$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

3. Types of Matrices

3.1 Row matrix :

If in a Matrix, there is only one row, then it is called a Row Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a row matrix if m = 1.

3.2 Column Matrix :

If in a Matrix, there is only one column, then it is called a Column Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a Column Matrix if n = 1.

3.3 Square Matrix :

If number of rows and number of column in a Matrix are equal, then it is called a Square Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a Square Matrix if m = n

Note :

(a) If m ≠ n then Matrix is called a Rectangular Matrix.

(b) The elements of a Square Matrix A for which i = j i.e. $a_{11}, a_{22}, a_{33}, ... a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or of leading diagonal of Matrix A.

(c) **Trance of a Matrix :** The sum of diagonal elements of a square matrix . A is called the trance of Matrix A which is denoted by tr A.

$$\text{tr } A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ...a_{nn}$$

3.4 Singleton Matrix :

If in a Matrix there is only one element then it is called Singleton Matrix. Thus

$A = [a_{ij}]_{m \times n}$ is a Singleton Matrix if m = n = 1.

3.5 Null or Zero Matrix :

If in a Matrix all the elements are zero then it is called a zero Matrix and it is generally denoted by O.

Thus $A = [a_{ij}]_{m \times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

3.6 Diagonal Matrix :

If all elements except the principal diagonal in a **Square Matrix** are zero, it is called a Diagonal Matrix. Thus a Square Matrix

$A = [a_{ij}]$ is a Diagonal Matrix if $a_{ij} = 0$, when i ≠ j

Note :

(a) No element of Principal Diagonal in diagonal Matrix is zero.

(b) Number of zero in a diagonal matrix is given by $n^2 - n$ where n is a order of the Matrix.

3.7 Scalar Matrix :

If all the elements of the diagonal of a **diagonal matrix** are equal , it is called a scalar matrix. Thus a Square Matrix $A = [a_{ij}]$ is a Scalar Matrix is

$$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases} \text{ where k is a constant.}$$

3.8 Unit Matrix :

If all elements of principal diagonal in a **Diagonal Matrix** are 1, then it is called Unit Matrix. A unit Matrix of order n is denoted by I_n.

Thus a square Matrix

$A = [a_{ij}]$ is a unit Matrix if

$$a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note :

Every unit Matrix is a Scalar Matrix.

3.9 Triangular Matrix :

A Square Matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero it is of two types-

(a) Upper Triangular Matrix : A Square Matrix $[a_{ij}]$ is called the upper triangular Matrix, if $a_{ij} = 0$ when $i > j$.

(b) Lower Triangular Matrix : A Square Matrix $[a_{ij}]$ is called the lower Triangular Matrix, if

$a_{ij} = 0$ when $i < j$

Note :

Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of Matrix.

3.10 Equal Matrix :

Two Matrix A and B are said to be equal Matrix if they are of same order and their corresponding elements are equal.

3.11 Singular Matrix :

Matrix A is said to be singular matrix if its determinant $|A| = 0$, otherwise non- singular matrix i.e.

If $\det |A| = 0 \Rightarrow$ Singular

and $\det |A| \neq 0 \Rightarrow$ non-singular

4. Addition and Subtraction of Matrices

If A $[a_{ij}]_{m \times n}$ and $[b_{ij}]_{m \times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum of corresponding element.

i.e. $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly their subtraction A – B is defined as

$$A - B = [a_{ij} - b_{ij}]_{m \times n}$$

Note :

Matrix addition and subtraction can be possible only when Matrices are of same order.

4.1 Properties of Matrices addition :

If A, B and C are Matrices of same order, then-

(i) $A + B = B + A$ (Commutative Law)

(ii) $(A+ B) + C = A + (B + C)$ (Associative Law)

(iii) $A + O = O + A = A$, where O is zero matrix which is additive identity of the matrix.

(iv) $A + (-A) = 0 = (-A) + A$ where $(-A)$ is obtained by changing the sign of every element of A which is additive inverse of the Matrix

(v) $\left.\begin{matrix} A + B = A + C \\ B + A = C + A \end{matrix}\right\} \Rightarrow B = C$ (Cancellation Law)

(vi) $\text{tr}(A \pm B) = \text{tr}(A) \pm \text{tr}(B)$

5. Scalar Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by

kA thus if $A = [a_{ij}]_{m \times n}$ then

$$kA = Ak = [ka_{ij}]_{m \times n}$$

5.1 Properties of Scalar Multiplication :

If A, B are Matrices of the same order and λ, μ are any two scalars then -

(i) $\lambda(A + B) = \lambda A + \lambda B$

(ii) $(\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda\mu) A = \mu(\lambda A)$

(iv) $(-\lambda A) = -(\lambda A) = \lambda(-A)$

(v) $\text{tr}(kA) = k\,\text{tr}(A)$

6. Multiplication of Matrices

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order $m \times p$, where

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

6.1 Properties of Matrix Multiplication :

If A, B and C are three matrices such that their product is defined , then

(i) $AB \neq BA$ (Generally not commutative)

(ii) (AB) C = A (BC) (Associative Law)

(iii) IA = A = AI

I is identity matrix for matrix multiplication

(iv) A (B + C) = AB + AC (Distributive Law)

(v) If $AB = AC \not\Rightarrow B = C$

(Cancellation Law is not applicable)

(vi) If AB = 0. It does not mean that A = 0 or B = 0, again product of two non- zero matrix may be zero matrix.

(vii) tr (AB) = tr (BA)

Note :

(i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$

(d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$

(e) $A(-B) = (-A)B = -(AB)$

6.2 Positive Integral powers of a Matrix :

The positive integral powers of a matrix A are defined only when A is a square matrix. Also then

$A^2 = A.A \quad A^3 = A.A.A = A^2A$

Also for any positive integers m,n

(i) $A^m A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I, I^m = I$

(iv) $A^o = I_n$ where A is a square matrices of order n.

7. Transpose of a Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix A and is denoted by A^T or A'.

From the definition it is obvious that

If order of A is $m \times n$, then order of A^T is $n \times m$.

7.1 Properties of Transpose :

(i) $(A^T)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(AB)^T = B^T A^T$

(iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 ......A_{n-1}A_n)^T$

$= A_n^T A_{n-1}^T .....A_3^T A_2^T A_1^T$

(vi) $I^T = I$

(vii) tr (A) = tr (A^T)

8. Symmetric & Skew-Symmetric Matrix

(a) Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i,j or $A^T = A$

Note :

(i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$.

(b) Skew - Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called

skew - symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j

or $A^T = -A$

Note :

(i) All Principal diagonal elements of a skew - symmetric matrix are always zero because for any diagonal element –

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

8.1 Properties of Symmetric and skew- symmetric matrices :

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is Skew-Symmetric Matrices.

(ii) If A is a Symmetric Matrix, then $-A$, KA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices where $n \in N$, $K \in R$ and B is a square matrix of order that of A.

(iii) If A is a skew symmetric matrix, then-

(a) A^{2n} is a symmetric matrix for $n \in N$

(b) A^{2n+1} is a skew-symmetric matrices for $n \in N$

(c) kA is also skew-symmetric matrix where $k \in R$.

(d) $B^T AB$ is also skew-symmetric matrix where B is a square matrix of order that of A

(iv) If A, B are two symmetric matrices, then-

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew - symmetric matrix.

(c) AB is a symmetric matrix when AB = BA.

(v) If A, B are two skew-symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(vi) If A is a skew - symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquelly be expressed as sum of a symmetric and skew symmetric matrix i.e.

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

9. Determinant of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by |A| or Det (A) is defined as

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

9.1 Properties of the Determinant of a matrix :

(i) |A| exists $\Leftrightarrow$ A is a square matrix

(ii) $|AB| = |A|\,|B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is a skew symmetric matrix of odd order then $|A| = 0$

(vii)If $A = \text{diag}(a_1, a_2 \ldots\ldots a_n)$ then $|A| = a_1 a_2 \ldots a_n$

(viii) $|A|^n = |A^n|$, $n \in N$.

10. Adjoint of a Matrix

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of matrix A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and F^{ij} be the cofactor of a_{ij} in |A|, then

$$\text{Adj } A = [F^{ij}]^T$$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ .... & .... & .... \\ .... & .... & .... \\ a_{n1} & a_{n2} & ...a_{nn} \end{bmatrix}$, then

$$\text{Adj } A = \begin{bmatrix} F_{11} & F_{12} & ...F_{1n} \\ F_{21} & F_{22} & ...F_{2n} \\ .... & .... & .... \\ .... & .... & .... \\ F_{n1} & F_{n2} & ...F_{nn} \end{bmatrix}^T$$

10.1 Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj}(A^T) = (\text{adj } A)^T$

(vi) adj (AB) = (adj B) (adj A)

(vii) $\text{adj}(A^m) = (\text{adj } A)^m,\ m \in N$

(viii) $\text{adj}(kA) = k^{n-1}(\text{adj } A),\ k \in R$

(ix) $\text{adj}(I_n) = I_n$

(x) adj 0 = 0

(xi) A is symmetric $\Rightarrow$ adj A is also symmetric

(xii) A is diagonal $\Rightarrow$ adj A is also diagonal

(xiii) A is triangular $\Rightarrow$ adj A is also triangular

(xiv) A is singular $\Rightarrow$ |adj A| = 0

11. Inverse of a Matrix

If A and B are two matrices such that

AB = I = BA

then B is called the inverse of A and it is denoted by A^{-1}, thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

To find inverse matrix of a given matrix A we use following formula

$$A^{-1} = \frac{\text{adj}A}{|A|}$$

Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

11.1 Properties of Inverse Matrix :

Let A and B are two invertible matrices of the same order, then

(i) $(A^T)^{-1} = (A^{-1})^T$

(ii) $(AB)^{-1} = B^{-1} A^{-1}$

(iii) $(A^k)^{-1} = (A^{-1})^k,\ k \in N$

(iv) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

(v) $(A^{-1})^{-1} = A$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) If $A = \text{diag}(a_1, a_2, ..., a_n)$, then

$A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, ..., a_n^{-1})$

(viii) A is symmetric matrix $\Rightarrow A^{-1}$ is symmetric matrix.

(ix) A is triangular matrix and $|A| \neq 0 \Rightarrow A^{-1}$ is a triangular matrix.

(x) A is scalar matrix $\Rightarrow A^{-1}$ is scalar matrix.

(xi) A is diagonal matrix $\Rightarrow A^{-1}$ is diagonal matrix.

(xii) $AB = AC \Rightarrow B = C$, iff $|A| \neq 0$.

SOLVED EXAMPLES

Ex.1 If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and a and b are arbitrary constants then $(aI + bA)^2 =$

(A) $a^2I + abA$ (B) $a^2I + 2abA$

(C) $a^2I + b^2A$ (D) None of these

Sol. Here $aI + bA = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$

$\therefore (aI + bA)^2 = \begin{pmatrix} a^2+0 & ab+ba \\ 0+0 & 0+a^2 \end{pmatrix}$

$= \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix} = a^2I + 2abA$ **Ans.[B]**

Ex.2 If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$, then which of the following statement is true ?

(A) $AB \neq AC$

(B) $AB = AC$

(C) $B \neq C \Rightarrow AB \neq AC$

(D) None of these

Sol. Here

$$AB = \begin{bmatrix} 1-6+2 & 4-3-4 & 1-3+2 & -3+4 \\ 2+2-3 & 8+1+6 & 2+1-3 & 1-6 \\ 4-6-1 & 16-3+2 & 4-1-3 & -3-2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix}$$

Also AC

$$= \begin{bmatrix} 2-9+4 & 1+6-10 & -1+3-2 & -2+3 \\ 4+3-6 & 2-2+15 & -2-1+3 & -4-1 \\ 8-9-2 & 4+6+5 & -4+3+1 & -8+3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -3 & 0 & 1 \\ 1 & 15 & 0 & -5 \\ -3 & 15 & 0 & -5 \end{bmatrix} = AB \ ;$$

Hence AC = AB is true **Ans. [B]**

Ex.3 If $A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$, $B = \begin{bmatrix} r & s \\ -s & r \end{bmatrix}$ then -

(A) AB = BA (B) $AB \neq BA$

(C) AB = –BA (D) None of these

Sol. Here $AB = \begin{bmatrix} pr-qs & ps+qr \\ -qr-ps & -qs+pr \end{bmatrix}$

Also $BA = \begin{bmatrix} rp-qs & qr+sp \\ -sp-qr & -qs+pr \end{bmatrix}$

Clearly AB = BA **Ans.[A]**

Ex.4 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then $A^2 - 4A =$

(A) 3I (B) 4I

(C) 5I (D) None of these

Sol. Here $A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore A^2 - 4A = \begin{bmatrix} 9-4 & 8-8 & 8-8 \\ 8-8 & 9-4 & 8-8 \\ 8-8 & 8-8 & 9-4 \end{bmatrix}$$

$$= 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5I$$ **Ans.[C]**

Ex. 5. If $f(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and if α, β, γ are angles of a triangle, then $f(\alpha).\ f(\beta).\ f(\gamma) =$

(A) I_2 (B) $-I_2$

(C) 0 (D) None of these

Sol. Hence

$$f(\alpha)\,f(\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

similarly

$$f(\alpha)\,f(\beta)\,f(\gamma) = \begin{bmatrix} \cos(\alpha+\beta+\gamma) & \sin(\alpha+\beta+\gamma) \\ -\sin(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{bmatrix} \text{ as } \alpha+\beta+\gamma = \pi$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -I_2.$$ **Ans.[B]**

Ex.6 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ then which of the following statements is true -

(A) AB = BA (B) $A^2 = B$

(C) $(AB)^T = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$ (D) None of these

Sol. We have $(AB)_{11} = 1.3 + 2.1 = 5$

$(BA)_{11} = 3.1 + 4.3 = 15$

$\therefore$ AB $\neq$ BA Again $(A^2)_{11} = 1.1 + 2.3$

$= 7 \neq 3 = (B)_{11}$

Also $(AB)^T = B^T A^T = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3+2 & 9+0 \\ 4+12 & 12+0 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 16 & 12 \end{bmatrix}$ is correct.

Ans.[C]

Ex.7 If $A = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}$ then which statement is true?

(A) $AA^T = I$ (B) $BB^T = I$

(C) AB $\neq$ BA (D) $(AB)^T = I$

Sol. Here $A\,A^T = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(BB^T)_{11} = (4)^2 + (1)^2 \neq 1$

$(AB)_{11} = 8 - 7 = 1$, $(BA)_{11} = 8 - 7 = 1$

$\therefore$ AB $\neq$ BA may be not true

Now

$AB = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}$

$= \begin{pmatrix} 8-7 & 2-2 \\ -28+28 & -7+8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$(AB)^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ **Ans.[D]**

Ex.8 If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, then |A| is equal to -

(A) 12 (B) –10

(C) 10 (D) 5

Sol. $|A| = \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = (4 \times 3 - 1 \times 2)$

$= 12 - 2 = 10$

$\left(\because \text{if } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{then} |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})\right)$

Ans.[C]

Ex.9. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix}$ then adj A is equal to -

(A) $\begin{bmatrix} -24 & 4 & 8 \\ 4 & 1 & 2 \\ 8 & 11 & -11 \end{bmatrix}$ (B) $\begin{bmatrix} -24 & 4 & 8 \\ 4 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$

(C) $\begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$ (D) None of these

Sol. Here $[A_{ij}] = \begin{bmatrix} \begin{vmatrix} 0 & 4 \\ 6 & 7 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 5 & 0 \\ 2 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 6 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} -24 & -27 & 30 \\ 4 & 1 & -2 \\ 8 & 11 & -10 \end{bmatrix}$ Hence transposing $[A_{ij}]$ we get

adj $A = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$ **Ans.[C]**

Ex.10 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ then adj (adj A) =

(A) $\begin{bmatrix} -18 & 36 & -54 \\ 36 & -54 & 18 \\ -54 & 18 & -36 \end{bmatrix}$

(B) $-\begin{bmatrix} 18 & 36 & 54 \\ 36 & 54 & 18 \\ 54 & 18 & 36 \end{bmatrix}$

(C) $18\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

(D) None of these

Sol. Hence we know adj (adj A) = $|A|^{n-2}$ A

Now if n = 3 then adj (adj A) = |A| A

$= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} A$

= {1(6–1) – 2 (4– 3) + 3 (2– 9)} A

= (5 – 2 – 21) A = – 18 A **Ans.[B]**

Ex.11 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ then A^{-n} is equal to-

(A) $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 \\ -n & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$ (D) None of these

Sol. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$A^{-2} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

$A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$ **Ans.[C]**

Ex.12 If A is idempotent and A + B = I, then which of the following is true?

(A) B is idempotent (B) AB = 0

(C) BA = 0 (D) All of these

Sol. Here A + B = I ⇒ B = I – A

Now B^2 = (I – A) (I – A)

= I^2 – AI – IA + A^2

= I – A – A + A^2

= I – A – A + A here A^2 = A since A is idempotent

= I – A = B

∴ B is idempotent is true

Again AB = A (I – A) = AI – A^2 = A – A =0

Also BA = (I – A) A = IA – A^2 = A – A =0

Hence all statements are true . **Ans.[D]**

Ex.13 If $k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is an orthogonal matrix then k is equal to -

(A) 1 (B) 1/2

(C) 1/3 (D) None of these

Sol. Here let

$A = k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$\therefore A^T = k\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Since A is orthogonal ∴ $AA^T = I$

$\Rightarrow k^2\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

$= k^2\begin{bmatrix} 1+4+4 & -2-2+4 & -2+4-2 \\ -2-2+4 & 4+1+4 & 4-2-2 \\ -2+4-2 & 4-2-2 & 4+4+1 \end{bmatrix}$

$= k^2\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9k^2 I$

$\Rightarrow 9k^2 = 1 \Rightarrow k^2 = \frac{1}{9} \Rightarrow k = \pm\frac{1}{3}$

Ans.[C]

Ex.14 If $A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and

$B = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$, and AB = 0,

then $\theta - \phi$ is equal to -

(A) 0

(B) even multiple of $(\pi / 2)$

(C) odd multiple of $(\pi / 2)$

(D) odd multiple of π

Sol. Here

$$AB = \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cos\phi\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos\theta\sin\theta\cos^2\phi + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ if } \cos(\theta - \phi) = 0$$

Now $\cos(\theta - \phi) = 0$, $\theta - \phi$ is an odd multiple of $(\pi/2)$.

Ans.[C]

Ex.15 If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then B equals -

(A) $I\cos\theta + J\sin\theta$ (B) $I\cos\theta - J\sin\theta$

(C) $I\sin\theta + J\cos\theta$ (D) $-I\cos\theta + J\sin\theta$

Sol. Here $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$= I\cos\theta + J\sin\theta$ **Ans.[A]**

Ex.16 If $M(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$M(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$ then

$[M(\alpha)\, M(\beta)]^{-1}$ is equals to -

(A) $M(\beta)\, M(\alpha)$ (B) $M(-\alpha)\, M(-\beta)$

(C) $M(-\beta)\, M(-\alpha)$ (D) $-M(\beta)\, M(\alpha)$

Sol. $[M(\alpha)\, M(\beta)]^{-1} = M(\beta)^{-1}\, M(\alpha)^{-1}$

Now $M(\alpha)^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} = M(-\alpha)$$

$M(\beta)^{-1} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix} = M(-\beta)$$

$\therefore [M(\alpha)\, M(\beta)]^{-1} = M(-\beta)\, M(-\alpha)$

Ans.[C]