DEFINITE INTEGRATION

(KEY CONCEPTS + SOLVED EXAMPLES)

DEFINITE INTEGRATION

1. *Definition*
2. *Properties of Definite Integral*
3. *Important Formulae*
4. *Summation of Series by Integration*

KEY CONCEPTS

1. Definition

If $\frac{d}{dx}[f(x)] = \phi(x)$ and a and b, are two values independent of variable x, then

$$\int_a^b \phi(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

is called **Definite Integral** of ϕ (x) within limits a and b. Here **a** is called the **lower limit** and **b** is called the **upper limit** of the integral. The interval [a, b] is known as **range of integration**. It should be noted that every definite integral has a unique value.

2. Properties of Definite Integral

[P-1] $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. the value of a definite integral remains unchanged if its variable is placed by any other symbol.

[P-2] $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e. the interchange of limits of a definite integral changes only its sign.

[P-3] $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where a < c < b.

or $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + ...+$

$\int_{c_n}^b f(x)\,dx$ where $a < c_1 < c_2 < ....c_n < b$.

Generally this property is used when the integrand has two or more rules in the integration interval.

[P-4] $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Note :

This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by a– x. With the help of above property following integrals can be obtained-

(i) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(ii) $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$

(iii) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(iv) $\int_0^1 f(\log x)\,dx = \int_0^1 f[\log(1-x)]\,dx$

(v) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx =$

$\int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx = \pi/4$

(vi) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}\,dx =$

$\int_0^{\pi/2} \frac{\cot^n}{1+\cot^n x}\,dx = \pi/4$

(vii) $\int_0^{\pi/2} \frac{1}{1+\tan^n x}\,dx =$

$\int_0^{\pi/2} \frac{1}{1+\cot^n x}\,dx = \pi/4$

(viii) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}\,dx =$

$\int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x + \sec^n x}\,dx = \pi/4$

(ix) $\int_0^{\pi/4} \log(1+\tan x)\,dx = (\pi/8)\log 2$

(x) $\int_0^{\pi/2} \log\cot x\,dx = \int_0^{\pi/2} \log\tan x\,dx = 0$

[P-5] $\int_{-a}^{a} f(x)\,dx$

$$= \begin{cases} 0, \text{ if } f(-x) = -f(x) \text{ i.e. if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, \text{ if } f(-x) = f(x) \text{ i.e. if } f(x) \text{ even} \end{cases}$$

This property is generally used when integrand is either even or odd function of x.

[P-6]

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x) \\ 0, \quad \text{if } f(2a-x) = -f(x) \end{cases}$$

It is generally used to make half the upper limit.

[P-7] If f(x) = f(x + a) , then

$$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$$

[P-8] $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

[P-9] $\frac{d}{dt}\left[\int_{\phi(t)}^{\psi(t)} f(x)\,dx\right] = f\{\psi(t)\}\psi'(t) - f\{\phi(t)\}\phi'(t)$

3. Some Important Formulae

I. $\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = -(\pi/2)\log 2.$

II. $\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Where Γ (n) is called **Gamma function** which satisfies the following properties

$\Gamma(n+1) = n\,\Gamma(n) = n!,\ \Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$

In place of gamma function, we can also use the following formula:

$\int_0^{\pi/2} \sin^m x \cos^n x\,dx$

$$= \frac{(m-1)(m-3)\ldots.(2 \text{ or } 1)(n-1)(n-3)\ldots.(2 \text{ or } 1)}{(m+n)(m+n-2)\ldots.(2 \text{ or } 1)} \times (1 \text{ or } \pi/2)$$

It is important to note that we multiply by (π/2) when both m and n are even.

III. Walli's formula :

(i) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\frac{2}{3}, & \text{when n is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}, & \text{when n is even} \end{cases}$$

4. Summation of Series by Integration

For finding sum of an infinite series with the help of definite integration, following formula is used-

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).\frac{1}{n} = \int_0^1 f(x)\,dx.$$

The following method is used to solve the questions on summation of series.

(i) After writing (r–1)th or rth term of the series, express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Therefore the given series will take the form

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).\frac{1}{n}$$

(ii) Now writing $\int$ in place of $\left(\lim_{n\to\infty}\sum\right)$, x in place of $\left(\frac{r}{n}\right)$ and dx in place of $\frac{1}{n}$, we get the integral $\int f(x)\,dx$ in place of above series.

(iii) The lower limit of this integral

$$= \lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=0}$$

where r = 0 is taken corresponding to first term of the series and upper limit

$$= \lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=n-1}$$

where r = n – 1 is taken corresponding to the last term.

SOLVED EXAMPLES

Ex.1 $\int_0^1 \frac{6x^2+1}{4x^3+2x+3}dx$ is equal to-

(A) $-\frac{1}{2}\log 3$ (B) $\frac{1}{2}\log 3$

(C) 2 log 3 (D) None of these

Sol. Let $4x^3 + 2x + 3 = t$ $\therefore$ $2(6x^2 + 1)dx = dt$

Limits –at x = 0; t = 3, at x = 1; t = 9

$\therefore I = \int_3^9 \frac{1}{2}\frac{dt}{t} = \frac{1}{2}[\log t]_3^9$

$= \frac{1}{2}[\log 9 - \log 3] = \frac{1}{2}\log 3$

Ans.[B]

Ex.2 $\int_0^1 \frac{x}{1+x^4}dx$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{8}$ (D) π

Sol. $I = \frac{1}{2}\int_0^1 \frac{2x}{1+(x^2)^2}dx$

$= \frac{1}{2}[\tan^{-1}x^2]_0^1$

$= \frac{1}{2}[\tan^{-1}1 - \tan^{-1}0]$

$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$

Ans.[C]

Ex.3 $\int_2^4 \frac{\sqrt{x^2-4}}{x}dx$ is equal to

(A) $2(3\sqrt{3} - \pi)$ (B) $2\sqrt{3} - \pi$

(C) $\frac{2}{3}(3\sqrt{3} - \pi)$ (D) π

Sol. Put x = 2 sec t, then

$I = \int_0^{\pi/3} \frac{2\tan t}{2\sec t}.2\sec t\tan t\,dt$

$= 2\int_0^{\pi/3} \tan^2 t\,dt$

$= 2\int_0^{\pi/3} (\sec^2 t - 1)dt = 2[\tan t - t]_0^{\pi/3}$

$= 2[\sqrt{3} - \pi/3] = \frac{2}{3}(3\sqrt{3} - \pi)$ **Ans. [C]**

Ex.4 $\int_0^{\pi^2/4} \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ is equal to

(A) 2 (B) 1

(C) $\pi/4$ (D) $\pi^2/8$

Sol. $\sqrt{x} = t$, $\frac{1}{\sqrt{x}}dx = 2dt$

$\therefore I = 2\int_0^{\pi/2} \sin t\,dt = 2(-\cos t)_0^{\pi/2} = 2(0 + 1) = 2$

Ans. [A]

Ex.5 If $f(x) = \begin{cases} 2x+1, & 0 < x < 1 \\ x^2+2, & 1 \le x < 2 \end{cases}$, then the value of $\int_0^2 f(x)dx$ is-

(A) $-\frac{19}{3}$ (B) $\frac{19}{3}$

(C) $\frac{3}{19}$ (D) None of these

Sol. $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$= \int_0^1 (2x+1)dx + \int_1^2 (x^2+2)dx$

$= [x^2 + x]_0^1 + \left[\frac{x^3}{3} + 2x\right]_1^2$

$= 2 - 0 + \left(\frac{20}{3} - \frac{7}{3}\right) = \frac{19}{3}$

Ans.[B]

Ex.6 $\int_0^{\pi/2} \log\sin x\,dx$ is equal to-

(A) $\frac{\pi}{2}\log 2$ (B) $-\frac{\pi}{2}\log 2$

(C) $\frac{\pi}{2}\log_{10}2$ (D) $-\frac{\pi}{2}\log_{10}2$

Sol. $I = \int_0^{\pi/2} \log\sin x\,dx$ ...(1)

$$I=\int_0^{\pi/2}\log\cos x\,dx \text{ (by p –4)} \qquad ...(2)$$

$$\therefore\ 2I=\int_0^{\pi/2}\log(\sin x\cos x)\,dx$$

$$=\int_0^{\pi/2}\log\left(\frac{\sin 2x}{2}\right)dx$$

$$=\int_0^{\pi/2}\log\sin 2x\,dx-\frac{\pi}{2}\log 2$$

$$=\frac{1}{2}\int_0^{\pi}\log\sin t\,dt-\frac{\pi}{2}\log 2,$$

where t = 2x

$$=2\,\frac{1}{2}\int_0^{\pi/2}\log\sin t\,dt-\frac{\pi}{2}\log 2 = 1-\frac{\pi}{2}\log 2$$

$$\Rightarrow I=-\frac{\pi}{2}\log 2$$

Ans.[B]

Ex.7 $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ is equal to

(A) π/2 (B) π/4

(C) π (D) 2π

Sol. Using prop. P–4, we have

$$I=\int_0^{\pi/2}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

Adding it to given integral we have

$$2I=\int_0^{\pi/2}dx = [x]_0^{\pi/2} = \pi/2$$

$\therefore$ I = π/4 **Ans.[B]**

Ex.8 If f(x) is an odd function of x, then $\int_{-\pi/2}^{\pi/2}f(\cos x)\,dx$ is equal to

(A) 0 (B) $\int_0^{\pi/2}f(\cos x)\,dx$

(C) $2\int_0^{\pi/2}f(\sin x)\,dx$ (D) $\int_0^{\pi}f(\cos x)\,dx$

Sol. Here f(cos x) will be even function of x,

$$I=\int_{-\pi/2}^{\pi/2}f(\cos x)\,dx = 2\int_0^{\pi/2}f(\cos x)\,dx$$

$$=2\int_0^{\pi/2}f(\sin x)\,dx$$

Ans.[C]

Ex.9 The value of the integral $\int_{-4}^{4}(ax^3+bx+c)\,dx$ depend on-

(A) b and c (B) a, b and c

(C) only c (D) a and c

Sol. $I=\int_{-4}^{4}(ax^3+bx)\,dx+\int_{-4}^{4}c\,dx$

$= 0 + 2\int_0^4 c\,dx$ (by P –5)

$= 2c[x]_0^4 = 8c$

Hence the value of I depends on c.

Ans.[C]

Ex.10 If $f(x)=\frac{x\cos x}{1+\sin^2 x}$, then $\int_{-\pi}^{\pi}f(x)\,dx$ equals-

(A) π/4 (B) π/2

(C) π (D) 0

Sol. Since $f(-x)=\frac{-x\cos(-x)}{1+\sin^2(\pi-x)}$

$$=\frac{-x\cos x}{1+\sin^2 x} = -f(x)$$

$$\therefore\ I=\int_{-\pi}^{\pi}f(x)\,dx = 0$$

Ans.[D]

Ex.11 $\int_0^{\pi/2}\sin^2 x\cos^3 x\,dx$ equals-

(A) 1 (B) 2/5

(C) 2/15 (D) 4/15

Sol. Using Walli's formula, we get

$$I = \frac{1.2}{5.3.1} = \frac{2}{15}$$

Ans.[C]

Ex.12 $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi$ equals-

(A) $\pi(\sqrt{2} - 1)$ (B) $\pi(\sqrt{2} + 1)$

(C) $\pi(2 - \sqrt{2})$ (D) None of these

Sol. $I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi$ ...(1)

$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin(\pi-\phi)} d\phi$

(by P – 8)

$= \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi} d\phi$ ...(2)

$2I = \int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi} d\phi = \pi \int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi} d\phi$

$= \pi [\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4} = 2\pi(\sqrt{2} - 1)$

$I = \pi(\sqrt{2} - 1)$

Ans.[A]

Ex.13 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to-

(A) 2 (B) – 2

(C) 1/2 (D) – 1/2

Sol. By property [P–8]

$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x(\pi - x)} = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$

Adding it with the given integral

$2I = \int_{\pi/4}^{3\pi/4} \frac{2dx}{1-\cos^2 x} = 2\int_{\pi/4}^{3\pi/4} \text{cosec}^2 x\, dx$

$= -2[\cot x]_{\pi/4}^{3\pi/4} = 4$

$\Rightarrow I = 2$ **Ans.[A]**

Ex.14 The value of $\int_0^{\pi/2} \sin^3 x\, dx$ is -

(A) 2/3 (B) 3/2 (C) 0 (D) $4\pi/3$

Sol. We have $I = \int_0^{\pi/2} \sin^3 x\, dx = \frac{(3-1)}{3} \cdot 1$

= 2/3.(Since n = 3 is odd).

Ans.[A]

Ex.15 $\lim_{n\to\infty} \left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + ... + \frac{1}{n}\right]$ is equal to-

(A) $\frac{\pi}{4} + \frac{1}{2}\log 2$ (B) $\frac{\pi}{4} - \frac{1}{2}\log 2$

(C) $\frac{\pi}{4} - 2\log\frac{1}{2}$ (D) None of these

Sol. $T_r = \frac{n+r}{n^2+r^2} = \frac{1}{n}\left[\frac{\left(1+\frac{r}{n}\right)}{1+\left(\frac{r}{n}\right)^2}\right]$

$\therefore$ given limit $= \int_0^1 \frac{1+x}{1+x^2} dx$

$= \left[\tan^{-1} x\right]_0^1 + \left[\frac{1}{2}\log(1+x^2)\right]_0^1$

$= \frac{\pi}{4} + \frac{1}{2}\log 2$

Ans.[A]

Ex.16 $\int_0^{\infty} \frac{x^3}{(1+x^2)^{9/2}} dx$ is equal to-

(A) 2/35 (B) 3/35

(C) 4/35 (D) None of these

Sol. Put x = tan t, then

$I = \int_0^{\pi/2} \frac{\tan^3 t}{\sec^9 t} \sec^2 t\, dt = \int_0^{\pi/2} \sin^3 t \cos^4 t\, dt$

$= \frac{2.3.1}{7.5.3.1} = \frac{2}{35}$

Ans.[A]

Ex.17 $\int_0^\infty \frac{dx}{1+e^x}$ is equal to-

(A) log 2 – 1 (B) log 2

(C) log 4 – 1 (D) –log 2

Sol. $I = \int_0^\infty \frac{e^{-x}}{e^{-x}+1}dx = -\left[\log(e^{-x}+1)\right]_0^\infty$

$= -[\log 1 - \log 2] = \log 2$

Ans.[B]

Ex.18 $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x}dx$ is equal to-

(A) 0 (B) 1

(C) $\pi/2$ (D) $\pi/4$

Sol. Using P–4, given integral becomes

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2-x)-\sin(\pi/2-x)}{1+\sin(\pi/2-x)\cos(\pi/2-x)}dx$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\cos x \sin x}dx = -I$$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

Ans.[A]

Ex.19 $\int_0^\infty \frac{x \log x}{(1+x^2)}dx$ equals

(A) 0 (B) log 7

(C) 5 log 13 (D) None of these

Sol. Here

$$\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx$$

$I = I_1 + I_2$

Putting $x = \frac{1}{t}$ in second integrand

$dx = -\frac{1}{t^2}dt$

$$\therefore I_2 = \int_1^0 \frac{\frac{1}{t}\log\left(\frac{1}{t}\right)}{\left(1+\frac{1}{t^2}\right)^2}\left(-\frac{1}{t^2}\right)dt$$

$$= -\int_0^1 \frac{t\log t}{(1+t^2)^2}dt = -I_1$$

$I = I_2 + I_1 = -I_1 + I_1 = 0$

Ans.[A]

Ex.20 $\int_0^\pi x\sin^4 x\,dx$ is equal to-

(A) $3\pi/16$ (B) $3\pi^2/16$

(C) $16\pi/3$ (D) $16\pi^2/3$

Sol. $I = \int_0^\pi x\sin^4 x\,dx$ ...(1)

$I = \int_0^\pi (\pi - x)\sin^4(\pi - x)dx$

$I = \int_0^\pi (\pi - x)\sin^4 x\,dx$ ...(2)

$\therefore 2I = \pi\int_0^\pi \sin^4 x\,dx$

$\Rightarrow 2I = \pi\int_0^\pi \sin^4 x\,dx$ [from property P-6]

$\Rightarrow I = \pi.\frac{3.1}{4.2}.\frac{\pi}{2} = \frac{3\pi^2}{16}$

Ans.[B]

Ex.21 $\int_1^2 \log x\,dx$ equals-

(A) 2 log 2 (B) $\log\left(\frac{2}{e}\right)$

(C) $\log\left(\frac{4}{e}\right)$ (D) None of these

Sol. $I = \int_1^2 1.\log x\,dx$ equals

(Intetgrating by parts by taking 1 as a second function)

$= \{x.\log x\}_1^2 - \int_1^2\left(\frac{1}{x}.\,x\right)dx$

$= (2\log 2 - 1\log 1) - [x]_1^2$

$= (2\log 2 - 0) - (2 - 1)$

$= \log 4 - \log e = \log\left(\frac{4}{e}\right)$

Ans.[C]

Ex.22 $\int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}\,dx$ equals-

(A) 2 (B) π

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Sol. $I = \int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}\,dx$

$I = \int_0^{\pi/2} \frac{2^{\sin(\pi/2-x)}}{2^{\sin(\pi/2-x)}+2^{\cos(\pi/2-x)}}\,dx$

$= \int \frac{2^{\cos x}}{2^{\cos x}+2^{\sin x}}\,dx$

$2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

Ans.[C]

Ex.23 $\int_0^x f(t) = x + \int_x^1 t\,f(t)\,dt$ then f(1) is equal to-

(A) $\frac{1}{2}$ (B) 0

(C) 1 (D) $-\frac{1}{2}$

Sol. $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$

$\Rightarrow$ f(x) = 1 + (0 – xf(x)) [diff. w.r.t. x]

$\Rightarrow$ f(x) = 1 – xf(x)

$\Rightarrow$ f(1) = 1– 1.f(x)

$\Rightarrow$ f(1) = $\frac{1}{2}$

Ans.[A]

Ex.24 If f(3 – x) = f(x) then $\int_1^2 x f(x)\,dx$ equals-

(A) $\frac{3}{2}\int_1^2 f(2-x)\,dx$ (B) $\frac{3}{2}\int_1^2 f(x)\,dx$

(C) $\frac{1}{2}\int_1^2 f(x)\,dx$ (D) None of these

Sol. Let x = 3 – y

$I = \int_2^1 (3-y)f(3-y)(-dy)$

$= \int_1^2 (3-x)f(3-x)\,dx$

$= \int_1^2 (3-x)f(x)\,dx$ [$\because$ f(3 – x) = f(x)]

$= 3\int_1^2 f(x)\,dx - I$

$I = \frac{3}{2}\int_1^2 f(x)\,dx$

Ans.[B]

Ex.25 $\int_0^1 \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$ is equal to-

(A) $\pi/2$ (B) $\pi/4$

(C) 0 (D) 1

Sol. Put $\sin^{-1} x = t$, $\frac{dx}{\sqrt{1-x^2}} = dt$ then

$\therefore I = \int_0^{\pi/2} t\sin t\,dt = [t(-\cos t)]_0^{\pi/2} + [\sin x]_0^{\pi/2} = 1$

Ans.[C]

Ex.26 The value of the integral $\int_0^{\pi/4} \frac{\sin\theta+\cos\theta}{9+16\sin 2\theta}\,d\theta$ is

(A) log 3 (B) log 2

(C) $\frac{1}{20}$ log 3 (D) $\frac{1}{20}$ log 2

Sol. Here

$I = \int_0^{\pi/4} \frac{\sin\theta+\cos\theta}{9+16(\sin 2\theta+1-1)}\,d\theta$

$= \int_0^{\pi/4} \frac{\sin\theta+\cos\theta}{25-16(1-\sin 2\theta)}\,d\theta$

$= \frac{1}{16}\int_0^{\pi/4} \frac{\sin\theta+\cos\theta}{(25/16)-(\sin\theta-\cos\theta)^2}\,d\theta$

$= \frac{1}{16}\int_{-1}^{0} \frac{dt}{(25/16)-t^2}$, where $(\sin\theta - \cos\theta) = t$

$= \frac{1}{16} \times \frac{1}{2\times 5/4}\left[\log\frac{(5/4)+t}{(5/4)-t}\right]_{-1}^{0}$

$= \frac{1}{40}\left[\log 1 - \log\frac{1/4}{9/4}\right] = \frac{1}{20}\log 3$

Ans.[C]

Ex.27 $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^2 x\,(\sin x + \cos x)\,dx$ is equal to-

(A) 2/15 (B) 4/15

(C) 2/5 (D) 8/15

Sol. $I = \int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x\,dx + \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^3 x\,dx$

(by P–5)

$= 0 + 2\int_{0}^{\pi/2} \sin^2 x \cos^3 x\,dx$

$= 2.\ \frac{1.2}{5.3.1} = \frac{4}{15}$

Ans.[B]

Ex.28 $\int_{0}^{2a} \frac{f(x)}{f(x)+f(2a-x)}\,dx$ is equal to-

(A) a (B) –a

(C) 0 (D) None of these

Sol. Using P–4, given integral becomes

$I = \int_{0}^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}\,dx$

Adding it with the given integral, we get

$2I = \int_{0}^{2a} 1\,dx = 2a \Rightarrow I = a$

Ans.[A]

Ex.29 $\int_{-1}^{3/2} |x \sin \pi x|\,dx$ is equal to

(A) $\frac{4}{\pi}$ (B) $\frac{3}{\pi} + \frac{1}{\pi^2}$

(C) $\frac{3}{\pi^2} + \frac{1}{\pi}$ (D) None of these

Sol. Obviously

$|x \sin \pi x| = \begin{cases} x \sin \pi x, & -1 < x < 1 \\ -x \sin \pi x, & 1 < x < 3/2 \end{cases}$

$\therefore\ I = \int_{-1}^{1} x \sin \pi x\,dx + \int_{1}^{3/2} (-x \sin \pi x)\,dx$

$= 2\int_{0}^{1} x \sin \pi x\,dx - \int_{1}^{3/2} x \sin \pi x\,dx$

$= 2\left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_0^1$

$- \left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_1^{3/2}$

$= 2\left(-\frac{\cos \pi}{\pi}\right) - \left(\frac{\sin(3\pi/2)}{\pi^2} + \frac{\cos \pi}{\pi}\right)$

$= \frac{3}{\pi} + \frac{1}{\pi^2}$ **Ans.[B]**

Ex.30 The value of $\int_{0}^{100\pi} \sqrt{1-\cos 2x}\,dx$ is

(A) $100\sqrt{2}$ (B) $200\sqrt{2}$

(C) $50\sqrt{2}$ (D) 0

Sol. $I = \sqrt{2}\int_{0}^{100\pi} |\sin x|\,dx$

$= 100\sqrt{2}\int_{0}^{\pi} |\sin x|\,dx$

$= 100\sqrt{2}\int_{0}^{\pi} \sin x\,dx = 100\sqrt{2}\,[-\cos x]_0^{\pi}$

$= 200\sqrt{2}$ **Ans.[B]**

Ex.31 $\int_{0}^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$ $(n \in N)$ is equal to-

(A) π^2 (B) $2\pi^2$

(C) π (D) 2π

Sol. $I = \int_{0}^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$= \int_{0}^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n} x(2\pi - x) + \cos^{2n}(2\pi - x)}\,dx$

(By P-4)

$= \int_{0}^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$\therefore\ 2I = 2\pi\int_{0}^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$$\Rightarrow I = 4\pi\int_0^{\pi/2}\frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$$

$= 4\pi(\pi/4) = \pi^2$.

Ans.[A]

Ex.32 $\displaystyle\int_0^{\pi/2}\frac{dx}{1+2\sin x+\cos x}$ equals-

(A) (1/2) log 3 (B) log 3

(C) (4/3) log 3 (D) None of these

Sol. Here

$$I = \int_0^{\pi/2}\frac{dx}{1+2\frac{2\tan(x/2)}{1+\tan^2(x/2)}+\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}}$$

$$= \int_0^{\pi/2}\frac{\sec^2(x/2)}{2\{1+2\tan(x/2)\}}dx$$

Let 1+ 2 tan (x/2) = t, then

$\sec^2(x/2)\,dx = dt$

$$\therefore I = \frac{1}{2}\int_1^3\frac{dt}{t} = \frac{1}{2}(\log t)_1^3$$

$$= \frac{1}{2}\log 3$$

Ans.[A]

Ex.33 $\displaystyle\int_0^{\pi/2}\frac{\sin 2x}{a\cos^2x+b\sin^2x}dx$ -

(A) $\frac{1}{b-a}\log\left(\frac{b}{a}\right)$ (B) $\frac{1}{b+a}\log\left(\frac{b}{a}\right)$

(C) $\frac{1}{b-a}\log\left(\frac{a}{b}\right)$ (D) $\frac{1}{b+a}\log\left(\frac{a}{b}\right)$

Sol. $$I = \left(\frac{1}{b-a}\right)\int_0^{\pi/2}\frac{(b-a)2\sin x\cos x}{a\cos^2x+b\sin^2x}dx$$

$$= \frac{1}{b-a}\left[\log(a\cos^2x+b\sin^2x)\right]_0^{\pi/2}$$

$$= \frac{1}{(b-a)}(\log b - \log a)$$

$$= \frac{1}{b-a}\log\left(\frac{b}{a}\right)$$

Ans.[A]

Ex.34 $\displaystyle\int_0^{\pi/2}(2\log\sin x-\log\sin 2x)\,dx$ equals-

(A) π log 2 (B) $-\pi$ log 2

(C) (π/2) log 2 (D) – (π/2) log 2

Sol. $$I = \int_0^{\pi/2}(2\log\sin x-\log 2\sin x\cos x)\,dx$$

$$= \int_0^{\pi/2}(2\log\sin x-\log 2-\log\sin x-\log\cos x)\,dx$$

$$= \int_0^{\pi/2}\log\sin x\,dx-\int_0^{\pi/2}\log 2dx-\int_0^{\pi/2}\log\cos x\,dx$$

= – (π/2) log 2. **Ans.[D]**

Ex.35 $\displaystyle\int_0^{\pi/2}\frac{x\sin x\cos x}{\sin^4x+\cos^4x}dx$ equals-

(A) $\frac{\pi^2}{8}$ (B) $\frac{\pi^2}{16}$

(C) $\frac{\pi^2}{4}$ (D) $\frac{\pi^2}{2}$

Sol. $$I = \frac{\pi}{4}\int_0^{\pi/2}\frac{\sin x\cos x}{\sin^4x+\cos^4x}dx$$

$$= \frac{\pi}{8}\int_0^{\pi/2}\frac{2\sin x\cos x}{(\sin^2x)^2+(1-\sin^2x)^2}dx$$

Assume $\sin^2x = t$

$\therefore$ 2sin x cos x dx = dt

$$\therefore I = \frac{\pi}{8}\int\frac{dt}{t^2+(1-t)^2}$$

$$I = \frac{\pi}{8}\int\frac{dt}{2t^2-2t+1}$$

$$= \frac{\pi}{16}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$= \frac{\pi}{16}\cdot\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\left[\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}\right]$$

$$= \frac{\pi}{8}[\tan^{-1}(2\sin^2 x - 1)]_0^{\pi/2}$$

$$= \frac{\pi}{8}[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= \frac{\pi}{8}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{16}$$

Ans.[B]

Ex. 36 $\int_0^{\pi/2} |\sin x - \cos x|\, dx$ equals-

(A) $2\sqrt{2}$ (B) $2(\sqrt{2}+1)$

(C) $2(\sqrt{2}-1)$ (D) 0

Sol. $\because$ |sin x – cos x|

$$= \begin{cases} -(\sin x - \cos x), & 0 < x < \pi/4 \\ (\sin x - \cos x), & \pi/4 < x < \pi/2 \end{cases}$$

$$\therefore I = \int_0^{\pi/4} -(\sin x - \cos x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$= [\cos x + \sin x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= 2\sqrt{2} - 2$$

Ans.[C]

Ex.37 The value of $\lim_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x}$ is-

(A) 0 (B) 1

(C) –1 (D) None of these

Sol. Let f(x) = $\int_0^x \cos t^2 dt$ and g(x) = x,

then f(0) = g(0) = 0

$$\therefore \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$\therefore \text{Given limit} = \lim_{x\to 0}\frac{\cos x^2 . 1 - \cos 0.0}{1}$$

$$\left[\text{since}\frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt = \int_{\phi(x)}^{\psi(x)} \frac{d}{dx}(f(t))\,dt\right.$$

$$\left. = f(\psi(x)\psi'(x) - f\{(\phi(x)\phi'(x)\}\right]$$

$\therefore$ Given limit

= cos 0 = 1. **Ans.[B]**

Ex.38 If n $\in$ Z, then

$$\int_0^{\pi} e^{\sin^2 x} \cos^3 (2n+1)x\; dx -$$

(A) –1 (B) 0

(C) 1 (D) π

Sol. Let f(x) = $e^{\sin^2 x}\cos^3(2n+1)x\; dx$

$\Rightarrow$ f(π – x) = $e^{\sin^2(\pi - x)}\cos^3(2n+1)(\pi - x)\,dx$

$$= -e^{\sin^2 x}\cos^3(2n+1)x$$

[$\because$ (2n + 1) is odd]

= – f(x)

So by P – 8, I = 0 **Ans.[B]**

Ex.39 The value of $\int_{-1/2}^{1/2}\left[\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2 - 2\right]^{1/2} dx$

equals-

(A) log (4/3) (B) 2 log (4/3)

(C) 4 log (4/3) (D) –4 log (4/3)

Sol. Here

$$I = \int_{-1/2}^{1/2}\left[\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2\right]^{1/2} dx$$

$$= \int_{-1/2}^{1/2}\left|\frac{4x}{x^2-1}\right| dx = 2\int_0^{1/2}\left|\frac{4x}{x^2-1}\right| dx$$

$$= 8\int_0^{1/2}\frac{x\,dx}{1-x^2} = -4\left[\log(1-x^2)\right]_0^{1/2}$$

$$= -4 \log\left(\frac{3}{4}\right) = 4 \log\left(\frac{4}{3}\right)$$ **Ans.[C]**

Ex.40 $\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$ equals-

(A) $\frac{\pi}{2}$ + log 2 (B) $\frac{\pi}{2}$ – log 2

(C) π – log 2 (D) None of these

Sol. $I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x-x^2}\right)dx$

$= \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$

$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]dx$

$= \int_0^1 \tan^{-1} x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$

$= 2\int_0^1 \tan^{-1} x\,dx$ [By prov. IV]

$= 2\left[x\tan^{-1} - \frac{1}{2}\log(1+x^2)\right]_0^1$

$= 2\frac{\pi}{4} - \log 2 = \frac{\pi}{2} - \log 2$

Ans.[B]

Ex.41 $\int_0^{\pi/4}(\sqrt{\tan x} + \sqrt{\cot x})\,dx$ is equal to-

(A) $\pi/2$ (B) $\pi/\sqrt{2}$

(C) $-\pi/2$ (D) $-\pi/\sqrt{2}$

Sol. Putting $\tan x = t^2$, then

$\sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{1+t^4}$

$\therefore I = \int_0^1 \left(t + \frac{1}{t}\right)\frac{2t\,dt}{1+t^4}$

$= 2\int_0^1 \frac{t^2+1}{t^4+1}dt = 2\int_0^1 \frac{1+1/t^2}{t^2+1/t^2}dt$

$= 2\int_0^1 \frac{d(t-1/t)}{(t-1/t)^2+2}$

$= \sqrt{2}\left[\tan^{-1}\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right)\right]_0^1$

$= \sqrt{2}\,[\tan^{-1}0 - \tan^{-1}(-\infty)]$

$= \sqrt{2}\,(\pi/2) = \pi/\sqrt{2}$

Ans.[B]

Ex.42 If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ is equal to-

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$

(C) $g(x)\,g(\pi)$ (D) $g(x)/g(\pi)$

Sol. $g(x+\pi) = \int_0^{\pi+x} \cos^4 t\,dt$

$= \int_0^{\pi} \cos^4 t\,dt + \int_{\pi}^{\pi+x} \cos^4 t\,dt$

[by P–3]

$= \int_0^{\pi} \cos^4 t\,dt + I_2$

Now in I_2, put $t = \pi + \theta$, then

$I_2 = \int_0^x \cos^4(\pi+\theta)\,d\theta$

$= \int_0^x \cos^4\theta\,d\theta = \int_0^x \cos^4 t\,dt$

$\therefore g(x+\pi) = \int_0^{\pi} \cos^4 t\,dt + \int_0^x \cos^4 t\,dt$

$= g(x) + g(\pi)$ **Ans.[A]**

Ex.43 $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x}$ is equal to-

(A) 0 (B) 2

(C) 1 (D) None of these

Sol. $I = \int_{-\pi/2}^{0} \frac{\cos x}{1+e^x}dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x}dx$

$= -\int_{\pi/2}^{0} \frac{\cos y}{1+e^{-y}}dy + \int_0^{\pi/2} \frac{\cos x}{1+e^x}$

(putting $x = -y$ in first integral)

$= \int_0^{\pi/2} \frac{e^y\cos y}{1+e^y}dy + \int_0^{\pi/2} \frac{\cos x}{1+e^x}dx$

$= \int_0^{\pi/2} \frac{e^x\cos x}{1+e^x}dx + \int_0^{\pi/2} \frac{\cos x}{1+e^x}dx$

$= \int_0^{\pi/2} \frac{(e^x+1)\cos x}{1+e^x}dx$

$= \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$

Ans.[C]

Ex.44 $\int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}dx$ is equal to-

(A) 0 (B) $2\int_0^1 \frac{\sin x}{3-|x|}dx$

(C) $\int_0^1 \frac{-2x^2}{3-|x|}dx$ (D) $2\int_0^1 \frac{\sin x - x^2}{3-|x|}dx$

Sol. $I = \int_{-1}^{1} \frac{\sin x - x^2}{3-|x|}dx$

$$= \int_{-1}^{1} \frac{\sin x}{3-|x|} dx - \int_{-1}^{1} \frac{x^2}{3-|x|} dx$$

$$= 0 - 2\int_{0}^{1} \frac{x^2}{3-|x|} dx$$

[$\because \frac{\sin x}{3-|x|}$ is an odd and $\frac{x^2}{3-|x|}$ is an even function]

$$= -2\int_{0}^{1} \frac{x^2}{3-|x|} dx$$

Ans.[C]

Ex.45 $\int_{-4}^{-5} e^{(x+5)^2} dx + 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$ is equal to-

(A) e^5 (B) e^4

(C) $3e^2$ (D) 0

Sol. Putting x = –t –4 in first integral and

$x = \frac{t}{3} + \frac{1}{3}$ in second integral

$$I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx = -\int_{0}^{1} e^{(-t+1)^2} dt = -\int_{0}^{1} e^{(t-1)^2} dt$$

$$I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$$

$$= 3\int_{0}^{1} e^{9(t/3-1/3)^2} dt = \int_{0}^{1} e^{(t-1)^2} dt$$

$\therefore$ I = $I_1 + I_2$ = 0. **Ans.[D]**

Ex.46 Let f be a positive function. If

$$I_1 = \int_{1-k}^{k} xf\{x(1-x)\}dx$$

$$I_2 = \int_{1-k}^{k} f[x(1-x)]dx$$

where $2k - 1 > 0$, then the value of I_1/I_2 is equal to-

(A) 2 (B) k

(C) 1/2 (D) 1

Sol. Using property P – 8, we have

$$I_1 = \int_{1-k}^{k} (k+1-k-x) f[(k+1-k-x) \times (1- k-1+ k + x)]dx$$

$$= \int_{1-k}^{k} (1-x) f[(1-x)(x)]dx$$

$$= \int_{1-k}^{k} f[x(1-x)]dx - \int_{1-k}^{k} xf[x(1-x)]dx$$

$= I_2 - I_1$

$\Rightarrow 2I_1 = I_2 \therefore \frac{I_1}{I_2} = \frac{1}{2}$

Ans.[C]